Eg: $X = [0,1)$, $\|x\| = |x|$,

$\langle x_n \rangle = \langle \frac{1}{2} - \frac{1}{n+1} \rangle \longrightarrow \frac{1}{2} - 0$ as $n \to \infty$

$\uparrow \qquad\qquad \in X$ ✓

Cauchy sequence & Convergent in $X = [0,1)$

But, $\Rightarrow$ $[0,1)$ is complete ~~...~~

$\langle y_n \rangle = \langle 1 - \frac{1}{n} \rangle \longrightarrow 1$ as $n \to \infty$

$\uparrow \qquad\qquad \notin X$.

Cauchy sequence but not cgt in $X = [0,1)$

So $X = [0,1)$ is not complete.

Examples:-

(1) Euclidean space $(\mathbb{R}^n, \|\cdot\|)$ is complete.

$$\|x\|_2 = \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{\frac{1}{2}} \quad \forall\ x = \langle \xi_j \rangle_{j=1}^{n} \in \mathbb{R}^n$$

$$d(x,y) = \|x-y\| = \left(\sum_{i=1}^{n} |\xi_i - \eta_i|^2\right)^{\frac{1}{2}}, \forall\ y = \langle \eta_i \rangle_{i=1}^{n} \in \mathbb{R}^n$$

metric d induced by $\|\cdot\|$

Let $\langle x_m \rangle_{m=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^n$.

where $x_m = (\xi_1^{(m)}, \xi_2^{(m)}, \dots, \xi_n^{(m)})$; $\xi_i^{(m)} \in \mathbb{R}$.

Then for each $\epsilon > 0$, $\exists$ a +ve integer $N(\epsilon)$ depending on ϵ s.t.

$$\left.\begin{aligned} & \|x_m - x_k\| < \epsilon \quad \forall\ m,k \geq N. \\ \Rightarrow\ & d(x_m, x_k) < \epsilon \quad \forall\ m,k \geq N. \\ \Rightarrow\ & \left(\sum_{j=1}^{n} |\xi_j^{(m)} - \xi_j^{(k)}|^2\right)^{1/2} < \epsilon \quad \forall\ m,k \geq N \end{aligned}\right\} \text{—(1)}$$

$$\Rightarrow \sum_{j=1}^{n} |\xi_j^{(m)} - \xi_j^{(k)}|^2 < \epsilon^2 \quad \forall\ m,k \geq N$$

$$\Rightarrow |\xi_j^{(m)} - \xi_j^{(k)}|^2 < \epsilon^2 \quad \forall\ m,k \geq N ;\ j = 1, \dots, n$$

$$\Rightarrow |\xi_j^{(m)} - \xi_j^{(k)}| < \epsilon \quad \forall\ m,k \geq N,\ j = 1, \dots, n. \quad \text{—}(*)$$

$\therefore$ for each fixed j, $\langle \xi_j^{(m)} \rangle$ is Cauchy sequence in $\mathbb{R}$ by $(*)$.